Quasinormal quantization in de Sitter spacetime

Quasinormal Quantization in deSitter Spacetime

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Abstract

A scalar field in four-dimensional deSitter spacetime (dS_4) has quasinormal modes which are singular on the past horizon of the south pole and decay exponentially towards the future. These are found to lie in two complex highest-weight representations of the dS_4 isometry group $\mathsf{SO}(4,1)$. The Klein-Gordon norm cannot be used for quantization of these modes because it diverges. However a modified 'R-norm', which involves reflection across the equator of a spatial S^3 slice, is nonsingular. The quasinormal modes are shown to provide a complete orthogonal basis with respect to the R-norm. Adopting the associated R-adjoint effectively transforms $\mathsf{SO}(4,1)$ to the symmetry group $\mathsf{SO}(3,2)$ of a 2+1-dimensional CFT. It is further shown that the conventional Euclidean vacuum may be defined as the state annihilated by half of the quasinormal modes, and the Euclidean Green function obtained from a simple mode sum. Quasinormal quantization contrasts with some conventional approaches in that it maintains manifest dS-invariance throughout. The results are expected to generalize to other dimensions and spins.

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1 Introduction

In this paper we present a new and potentially useful approach to an old problem: the quantization of a scalar field in four-dimensional de Sitter spacetime (dS_4), which has an $\mathsf{SO}(4,1)$ isometry group. One standard approach begins with the spherical harmonics of the S^3 spatial sections, and proceeds by solving the wave equation for the time-dependent modes. Linear combinations of these modes that are nonsingular under a certain analytic continuation are then identified as the Euclidean modes and used to define the quantum Euclidean vacuum. The vacuum so constructed exhibits manifest $\mathsf{SO}(4)$-invariance and can also be shown to possess the full $\mathsf{SO}(4,1)$ symmetry of dS_4. Another common approach singles out the southern causal diamond and relies on a special Killing vector field, denoted L_0, which generates southern Killing time and whose corresponding eigenmodes have real frequency ω. This construction displays manifest $\mathsf{SO}(3) \times \mathsf{SO}(1,1)$ symmetry and again leads to the dS-invariant Euclidean vacuum. The modes employed in these and similar constructions are *not* in $\mathsf{SO}(4,1)$ multiplets and hence $\mathsf{SO}(4,1)$-invariance of the final expressions is nontrivial.

For example, the action of the dS_4 isometries on the southern diamond L_0 eigenmodes shifts the frequency by *imaginary* integer multiples of $2\pi/\ell$ (where ℓ is the dS radius) while the usual southern diamond modes all have real frequencies.

It is natural to adopt scalar modes which lie in highest-weight representations of $\mathsf{SO}(4,1)$ and therefore boast manifest dS-invariance. These turn out to be nothing but the quasinormal modes of the southern diamond, which have complex L_0 eigenvalues and comprise four real or two complex highest-weight representations.* They are singular on the past horizon and decay exponentially towards the future, as opposed to the conventional southern diamond modes which oscillate everywhere. In order to quantize in a quasinormal mode basis, a norm is needed. The singularities on the past horizon render the Klein-Gordon norm singular, which is presumably why the quasinormal modes have not typically been used for quantization. However a variety of other equally suitable norms *have* been employed for various reasons in dS [5, 6, 7, 8, 9, 10, 11, 12]. One of them – the so-called R-norm [11] – differs from the Klein-Gordon norm by the insertion of a spatial reflection through the equator of the S^3 slice, thereby exchanging the north and south poles. We demonstrate that the R-norm is finite for quasinormal modes and hence suitable for quantization. We also show that the Euclidean vacuum has the simple and manifestly dS-invariant definition as the state annihilated by two of the four sets of quasinormal modes. Moreover the Euclidean Green function is shown, as anticipated in [13], to be obtainable from a simple sum over the quasinormal modes. We caution the reader that quasinormal modes have singularities on the past horizon which we regulate with an $i\epsilon$-prescription. Our statements about completeness and mode sums depend on taking the $\epsilon \to 0$ limit at the end of our calculations.

The real Killing vectors which generate the dS isometries have an $\mathsf{SO}(4,1)$ Lie bracket algebra and are antihermitian with respect to the Klein-Gordon norm. However they have mixed hermiticity under the R-norm. Multiplication by appropriate factors of i produces complex Killing vector fields which are antihermitian under the R-norm. The Lie algebra of these R-antihermitian vector fields turns out to generate $\mathsf{SO}(3,2)$, which is precisely the symmetry group of a 2+1-dimensional CFT, and the transformed notion of hermiticity is exactly the one conventionally employed when studying CFT_3 on the Euclidean plane [9]. Hence this $\mathsf{SO}(4,1) \to \mathsf{SO}(3,2)$ transformation, and the use of quasinormal modes, fits naturally within the dS_4/CFT_3 correspondence [7, 14, 15, 16, 17, 18, 19].†

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*Interesting work on the normalizability and completeness of quasinormal modes for black holes can be found in [1, 2, 3, 4].

†For every bulk scalar Φ one expects a dual operator $\mathcal{O}$ in the boundary CFT_3. The bulk state with one quantum in the lowest quasinormal mode is dual to the CFT_3 state associated to an $\mathcal{O}$ insertion at the north pole of the S^3 at $\mathcal{I}^+$, and the descendants fill out $\mathsf{SO}(3,2)$ representations on both sides of the duality [11].

This paper is organized as follows. In section 2, we begin by reviewing the standard global S^3 modes and the construction of Euclidean modes and Green functions. In section 3, we show that the quasinormal modes comprise the highest-weight modes and their descendants, specializing for simplicity to the case of conformal mass $m^2\ell^2 = 2$. Then in section 4, the modified R-norm and its properties are presented. Next, in section 5 we prove that half the quasinormal modes are Euclidean modes and demonstrate their completeness by deriving the Euclidean Green function from a quasinormal mode sum. In section 6 we generalize these results to the case of light scalars with $m^2\ell^2 \leq 9/4$. Finally, in section 7 we isolate quasinormal modes that vanish in the northern or southern diamonds – the analogues of Rindler modes in Minkowski space. These might eventually be useful for understanding the thermal nature of physics in a single dS causal diamond, but we do not pursue this direction further herein.

In addition, in Appendix A we provide the explicit forms of dS_4 Killing vectors as well as their commutation relations. This is followed by Appendix B, which computes the norm of spherically symmetric descendants using the $\mathsf{SO}(4,1)$ algebra, and Appendix C, which provides details on the Euclidean two-point function evaluated on the south pole observer's worldline.

We expect our discussion to generalize to the case of heavy scalars with $m^2\ell^2 > 9/4$ as well as other dimensions and spin.

2 $\mathsf{SO}(4)$-invariant global mode decomposition

In this section we describe the standard dS_4 mode decomposition in terms of the spherical harmonics of the S^3 spatial sections. These modes are regular everywhere on dS_4 and sometimes referred to as 'global modes'.

We will work in the dS_4 global coordinates $x = (t, \psi, \theta, \phi)$ with line element

$$\frac{ds^2}{\ell^2} = -dt^2 + \cosh^2 t\, d\Omega_3^2 = -dt^2 + \cosh^2 t\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right], \tag{2.1}$$

where $\Omega = (\psi, \theta, \phi)$ are coordinates on the global S^3 slices. We denote the north and south pole by

$$\Omega_{SP} \sim \psi = 0, \qquad \Omega_{NP} \sim \psi = \pi. \tag{2.2}$$

In this coordinate system, the dS-invariant distance function $P(x; x')$ is given by

$$P(t, \Omega; t', \Omega') = \cosh t \cosh t' \cos\Theta_3(\Omega, \Omega') - \sinh t \sinh t', \tag{2.3}$$

where $\Theta_3(\Omega, \Omega')$ denotes the geodesic distance function on S^3 and

$$\cos \Theta_3(\Omega, \Omega') = \cos\psi \cos\psi' + \sin\psi \sin\psi' \left[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')\right]. \tag{2.4}$$

Following the notation of [11], solutions of the wave equation

$$\left(\nabla^2 - m^2\right)\Phi = 0 \tag{2.5}$$

may be expanded in representations of the $\mathsf{SO}(4)$ rotations of the S^3 spatial slice at fixed t:

$$\Phi_{Lj}(x) = y_L(t) Y_{Lj}(\Omega). \tag{2.6}$$

These have total $\mathsf{SO}(4)$ angular momentum L and spin labeled by the multi-index j. The S^3 spherical harmonics Y_{Lj} obey the identities

$$\begin{aligned}
Y^*_{Lj}(\Omega) &= (-)^L Y_{Lj}(\Omega) \; = \; Y_{Lj}(\Omega_A), \\
D^2 Y_{Lj}(\Omega) &= -L(L+2)Y_{Lj}(\Omega), \\
\int_{S^3} d^3\Omega\sqrt{h}\; Y^*_{Lj}(\Omega)Y_{L'j'}(\Omega) &= \delta_{L,L'}\delta_{j,j'}, \\
\sum Y^*_{Lj}(\Omega)Y_{Lj}(\Omega') &= \frac{1}{\sqrt{h}}\delta^3(\Omega - \Omega'),
\end{aligned} \tag{2.7}$$

where Ω_A denotes the antipodal point of Ω, while $\sqrt{h} = \sin^2\psi\sin\theta$ and D^2 are the measure and Laplacian on the unit S^3, respectively. Here and hereafter, $\sum$ denotes summation over all allowed values of L and the multi-index j. The time dependence $y_L(t)$ is then governed by the differential equation

$$\partial_t^2 y_L + 3\tanh t\,\partial_t y_L + \left[m^2\ell^2 + \frac{L(L+2)}{\cosh^2 t}\right] y_L = 0. \tag{2.8}$$

The general solution has the $\mathcal{I}^+$ falloff

$$y_L \to e^{-h_\pm t}, \qquad h_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} - m^2\ell^2}. \tag{2.9}$$

For the time being, we restrict our attention to the case $m^2\ell^2 = 2$, which corresponds to a conformally coupled scalar with

$$h_+ = 2, \qquad h_- = 1. \tag{2.10}$$

The case of generic mass is qualitatively similar but with algebraic functions replaced by hypergeometric ones. We give the correspondingly more involved formulae in section 6.

The so-called Euclidean modes, which define the vacuum, are those which remain nonsingular on the southern hemisphere when dS_4 is analytically continued to S^4. In other words, they are defined by the condition

$$y_L^E\left(t=-\frac{i\pi}{2}\right)=\text{nonsingular}. \tag{2.11}$$

Explicitly, these modes are [11]:

$$y_L^E=\frac{2^{L+1}}{\sqrt{2L+2}}\frac{\cosh^L t\, e^{-(L+1)t}}{(1-ie^{-t})^{2L+2}}. \tag{2.12}$$

Note that they are singular on the northern hemisphere at $t=i\pi/2$. In terms of the Klein-Gordon inner product on global S^3 slices,

$$\langle\Phi_1,\Phi_2\rangle_{KG}\equiv i\int_{S^3}d^3\Sigma^\mu\ \Phi_1^*\overleftrightarrow{\partial_\mu}\Phi_2, \tag{2.13}$$

we have normalized the modes such that

$$\left\langle\Phi_{Lj}^E,\Phi_{L'j'}^E\right\rangle_{KG}=\delta_{LL',jj'}. \tag{2.14}$$

Using these modes, one can define the Euclidean vacuum by the condition

$$\left\langle\Phi_{Lj}^E,\hat\Phi\right\rangle_{KG}|0_E\rangle=0, \tag{2.15}$$

where $\hat\Phi$ is the quantum field operator. Since the modes Φ_{Lj}^E are not $\mathsf{SO}(4,1)$-invariant, it is not immediately obvious that the Euclidean vacuum is dS-invariant, but this can be checked explicitly. The Wightman function is

$$G_E(x;x')\equiv\langle 0_E|\,\hat\Phi(x)\hat\Phi(x')\,|0_E\rangle=\sum\Phi_{Lj}^E(x)\Phi_{Lj}^{E*}(x'). \tag{2.16}$$

Using the $i\epsilon$-prescription, this may be expressed in terms of the dS-invariant distance function $P(x;x')$ as

$$G_E(x;x')=\frac{1}{8\pi^2}\frac{1}{1-P(x;x')+is(x;x')\epsilon}, \tag{2.17}$$

where $s(x;x')>0$ if x lies in the future of x' and $s(x;x')<0$ otherwise.

If we rewrite $P(x;x')$ in terms of the coordinates X on the embedding 5D manifold with Minkowski spacetime metric η (in which dS_4 is just the hyperboloid $\eta_{\mu\nu}X^\mu X^\nu = \ell^2$), then we can represent $s(x;x')$ by [20]

$$s(X;Y) \equiv X^0 - Y^0. \tag{2.18}$$

Note that this is exactly the same as sending $X^0 - Y^0 \to X^0 - Y^0 - i\epsilon$, since this latter choice of $i\epsilon$-prescription shifts $P(X;Y) = \eta_{\mu\nu}X^\mu Y^\nu/\ell^2$ as follows:

$$P(X;Y) \to P(X;Y) - i\epsilon(X^0 - Y^0). \tag{2.19}$$

dS_4 has 10 real Killing vectors which, letting $k \in \{1,2,3\}$, we will refer to as the dilation L_0, the 3 boosts $M_k - M_{-k}$ and the 6 $\mathsf{SO}(4)$ rotation generators J_k and $M_k + M_{-k}$. Their explicit forms are given in Appendix A. The global modes indexed by L transform in the (L,L) representation of $\mathsf{SO}(4)$ with quadratic Casimir $L(L+2)$, but they are not in definite $\mathsf{SO}(4,1)$ representations. In particular, acting arbitrarily many times with the L_0 raising or lowering operators $M_{\pm k}$ gives a nonzero result, so they are in representations with unbounded L_0. In the next section we discuss a dS_4 mode decomposition using the highest-weight representations of $\mathsf{SO}(4,1)$.

3 $\mathsf{SO}(4,1)$-invariant quasinormal modes

In this section we describe the $\mathsf{SO}(4,1)$-invariant mode decomposition in terms of (anti-) quasinormal modes. We begin by defining

$$\begin{aligned} G_\pm(x;x') &\equiv G_E(x;x') \pm G_E(x;x'_A) \\ &= \frac{1}{8\pi^2}\left[\frac{1}{1-P(x;x')+i(X^0-X'^0)\epsilon} \pm \frac{1}{1+P(x;x')+i(X^0+X'^0)\epsilon}\right], \end{aligned} \tag{3.1}$$

where x_A denotes the antipodal point of x. These Green functions fall off like $e^{-2h_\pm t}$ as both arguments are taken to $\mathcal{I}^+$. Next we introduce 'Ω-modes' as follows:

$$\Phi^\pm_\Omega(x) \equiv \frac{\pi}{\sqrt{h_\pm}} \lim_{t\to\infty} e^{h_\pm t} G_\pm(x;\Omega,t). \tag{3.2}$$

The normalization factor was chosen for future convenience.

In terms of the global coordinates, the Ω-modes take the explicit form

$$\begin{aligned}
\Phi_\Omega^-(x) &= \frac{1}{2\pi}\frac{1}{[\sinh t - i\epsilon - \cosh(t)\cos\Theta_3(\Omega,x)]} \qquad (3.3)\\
&= \frac{1}{2\pi}\frac{1}{[\sinh t - \cosh t\cos\Theta_3(\Omega,x)]} - \frac{i}{2\cosh t}\delta(\tanh t - \cos\Theta_3(\Omega,x)),\\
\Phi_\Omega^+(x) &= -\frac{1}{\sqrt{2}\pi}\frac{1}{[\sinh t - i\epsilon - \cosh(t)\cos\Theta_3(\Omega,x)]^2} \qquad (3.4)\\
&= -\frac{1}{\sqrt{2}\pi}\frac{1}{[\sinh t - \cosh t\cos\Theta_3(\Omega,x)]^2} - \frac{i}{\sqrt{2}\cosh^2 t}\delta'(\tanh t - \cos\Theta_3(\Omega,x)).
\end{aligned}$$

The delta-functions above are normalized as one-dimensional delta-functions, that is, such that $\int_{-\infty}^{\infty} dy\, \delta(y) = 1$. The Ω-modes can be expanded in terms of the Euclidean global $\mathsf{SO}(4)$ modes as follows:

$$\begin{aligned}
\Phi_\Omega^-(x) &= \sqrt{8}\pi\sum\left[\frac{1}{\sqrt{L+1}}Y^*_{Lj}(\Omega)\right]\Phi^E_{Lj}(x),\\
\Phi_\Omega^+(x) &= -i\sqrt{16}\pi\sum\left[\sqrt{L+1}Y^*_{Lj}(\Omega)\right]\Phi^E_{Lj}(x).
\end{aligned} \qquad (3.5)$$

The lowest-weight and highest-weight modes are respectively given by [11]

$$\Phi^\pm_{lw}(x) \equiv \Phi^\pm_{\Omega_{SP}}(x), \qquad \Phi^\pm_{hw}(x) \equiv \Phi^\pm_{\Omega_{NP}}(x). \qquad (3.6)$$

By construction, the modes $\Phi^\pm_{hw}$ are eigenfunctions of L_0 with eigenvalues $-h_\pm$ and are annihilated by M_{-k} for each $k \in \{1,2,3\}$. The descendants of the highest-weight modes are obtained by acting with the M_{+k}, for any $k \in \{1,2,3\}$ (see Appendix A):

$$M_{+K}\Phi^\pm_{hw}(x) \equiv M_{+k_1}\cdots M_{+k_n}\Phi^\pm_{hw}(x), \qquad (3.7)$$

where K is a multi-index denoting the set $\{k_1,\ldots,k_n\}$.

The southern causal diamond (sometimes called the static patch) is the intersection of the causal past and future of the south pole. The highest-weight states are smooth everywhere in this diamond except for the past horizon where they are singular, and they decay exponentially towards the future. Therefore they, together with all their descendants appearing in (3.7) and their complex conjugates, comprise the quasinormal modes of the southern diamond. The lowest-weight states (with their descendants and complex conjugates) are singular on the future horizon and are the antiquasinormal modes of the southern diamond.

To emphasize this we adopt the notation

$$\Phi^{\pm}_{QN}(x) \equiv \Phi^{\pm}_{hw}(x) = \Phi^{\pm}_{\Omega_{NP}}(x), \qquad \Phi^{\pm}_{AQN}(x) \equiv \Phi^{\pm}_{lw}(x) = \Phi^{\pm}_{\Omega_{SP}}(x). \tag{3.8}$$

At this point we have eight highest-weight representations of $\mathsf{SO}(4,1)$, with elements

$$\begin{array}{llll} M_{+K}\Phi^{+}_{QN}, & M_{+K}\Phi^{-}_{QN}, & M_{+K}\Phi^{+*}_{QN}, & M_{+K}\Phi^{-*}_{QN}, \\ M_{-K}\Phi^{+}_{AQN}, & M_{-K}\Phi^{-}_{AQN}, & M_{-K}\Phi^{+*}_{AQN}, & M_{-K}\Phi^{-*}_{AQN}. \end{array} \tag{3.9}$$

We shall see below that this is an overcomplete set: only the first or second row of modes is needed to obtain a complete basis.

4 R-norm

We wish to expand the scalar field operator in the (anti-)quasinormal modes. Towards this end it is useful to introduce an inner product. The Klein-Gordon norms of the Ω-modes are

$$\begin{aligned} \left\langle \Phi^{\pm}_{\Omega_1}, \Phi^{\pm}_{\Omega_2} \right\rangle_{KG} &= \mp \frac{16\pi^2}{h_{\pm}} \Delta_{\pm}(\Omega_1, \Omega_2), \\ \left\langle \Phi^{+}_{\Omega_1}, \Phi^{-}_{\Omega_2} \right\rangle_{KG} &= -\left\langle \Phi^{-}_{\Omega_1}, \Phi^{+}_{\Omega_2} \right\rangle_{KG} = \frac{16\pi^2}{\sqrt{2}} \frac{i}{\sqrt{h}} \delta^3(\Omega_1 - \Omega_2), \end{aligned} \tag{4.1}$$

where

$$\Delta_{\pm}(\Omega, \Omega') = \frac{1}{2^{2\mp 1}\pi^2} \frac{1}{(1-\cos\Theta_3)^{h_{\pm}}} \tag{4.2}$$

denote the two-point functions for a CFT_3 operators with dimensions $h_{\pm}$. These satisfy

$$-\int d^3\Omega'' \sqrt{h}\ \Delta_{+}(\Omega, \Omega'') \Delta_{-}(\Omega'', \Omega') = \frac{1}{\sqrt{h}} \delta^3(\Omega - \Omega'). \tag{4.3}$$

The norm of a highest-weight quasinormal mode is obtained by setting $\Omega_1 = \Omega_2 = \Omega_{NP}$, which is evidently divergent. Hence the Klein-Gordon norm is not suitable for quantization of the quasinormal modes.

Alternate norms have been employed in de Sitter spacetime for a variety of reasons [7, 8, 9, 10, 11]. Here, following [11], a useful 'R-norm' can be defined by inserting a reflection R on S^3 across the equator:

$$\begin{aligned} R : (\psi, \theta, \phi) &\to (\pi - \psi, \theta, \phi), \\ \langle \Phi_1, \Phi_2 \rangle_R &\equiv \langle \Phi_1, R\Phi_2 \rangle_{KG}. \end{aligned} \tag{4.4}$$

With respect to this R-norm,

$$\left\langle \Phi^{\pm}_{\Omega_1}, \Phi^{\pm}_{\Omega_2} \right\rangle_R = \mp \frac{16\pi^2}{h_\pm} \Delta_\pm(\Omega_1, R\Omega_2). \tag{4.5}$$

In particular, the norms of the highest-weight quasinormal modes are simply

$$\left\langle \Phi^{+}_{QN}, \Phi^{+}_{QN} \right\rangle_R = -1, \qquad \left\langle \Phi^{-}_{QN}, \Phi^{-}_{QN} \right\rangle_R = 1, \qquad \left\langle \Phi^{+}_{QN}, \Phi^{-}_{QN} \right\rangle_R = 0, \tag{4.6}$$

while the R-inner product between a quasinormal mode and the complex conjugate of any quasinormal mode vanishes.

Changing the norm affects the hermiticity properties of the 10 real Killing vector fields which generate the dS_4 isometries. Under the Klein-Gordon norm, their adjoints are

$$\langle L_0 f, g \rangle_{KG} = \langle f, -L_0 g \rangle_{KG}\,, \quad \langle J_k f, g \rangle_{KG} = \langle f, -J_k g \rangle_{KG}\,, \quad \langle M_{\mp k} f, g \rangle_{KG} = \langle f, -M_{\mp k} g \rangle_{KG}\,, \tag{4.7}$$

so that the Killing generators are all antihermitian. However, under the modified R-norm,

$$\langle L_0 f, g \rangle_R = \langle f, L_0 g \rangle_R\,, \qquad \langle J_k f, g \rangle_R = \langle f, -J_k g \rangle_R\,, \qquad \langle M_{\mp k} f, g \rangle_R = \langle f, M_{\pm k} g \rangle_R\,. \tag{4.8}$$

To recover antihermitian generators in the R-norm, one must send $M_k + M_{-k} \to i(M_k + M_{-k})$ and $L_0 \to iL_0$ while keeping the rest of the generators the same. The Lie bracket algebra of the antihermitian vector fields is then $\mathsf{SO}(3,2)$ rather than $\mathsf{SO}(4,1)$. See Appendix A for more details.

Interestingly $\mathsf{SO}(3,2)$ is the symmetry group of a CFT in $2+1$ dimensions. This suggests that the quantum states on which these generators act could belong to a $2+1$-dimensional CFT, which fits in nicely with the dS_4/CFT_3 conjecture.

Using (4.8) we can compute the norm of the descendants. For example, the norm of the first descendant is (not summing over k)

$$\left\langle M_{+k}\Phi^{\pm}_{QN}, M_{+k}\Phi^{\pm}_{QN} \right\rangle_R = \left\langle \Phi^{\pm}_{QN}, M_{-k}M_{+k}\Phi^{\pm}_{QN} \right\rangle_R = 2h_\pm \left\langle \Phi^{\pm}_{QN}, \Phi^{\pm}_{QN} \right\rangle_R. \tag{4.9}$$

Observe that under this R-norm, the descendants of Φ^{+}_{QN} are orthogonal to those of Φ^{-}_{QN}. For the $\mathsf{SO}(3)$-symmetric states, we provide the exact formula in Appendix B.

5 Completeness of quasinormal modes

In this section we show that the quasinormal modes

$$\left\{M_{+K}\Phi^{-}_{QN}, \qquad M_{+K}\Phi^{-*}_{QN}, \qquad M_{+K}\Phi^{+}_{QN}, \qquad M_{+K}\Phi^{+*}_{QN}\right\} \tag{5.1}$$

form a complete set in the sense that the Euclidean Green function can be written as a simple sum over such modes. In particular, the antiquasinormal modes are not needed.

First we note from (3.5) that the quasinormal modes can be written as linear combinations of the global Euclidean modes, without using their complex conjugates. Therefore they are themselves Euclidean modes, and the Euclidean vacuum obeys

$$\left\langle M_{+K}\Phi^{\pm}_{QN}, \hat{\Phi}\right\rangle_R |0_E\rangle = 0. \tag{5.2}$$

Note that this relation, unlike the corresponding one for the global Euclidean modes, is manifestly dS-invariant because the quasinormal modes lie in representations of $\mathsf{SO}(4,1)$.

Let us now assume that we can expand the field operator in the presumably complete basis (5.1):

$$\begin{aligned}\hat{\Phi} = \sum_{K,K'} \Big(& N^{+}_{KK'} \left\langle M_{+K}\Phi^{+}_{QN}, \hat{\Phi}\right\rangle_R M_{+K'}\Phi^{+}_{QN} - N^{+}_{K'K} \left\langle M_{+K}\Phi^{+*}_{QN}, \hat{\Phi}\right\rangle_R M_{+K'}\Phi^{+*}_{QN} \\ & + N^{-}_{KK'} \left\langle M_{+K}\Phi^{-}_{QN}, \hat{\Phi}\right\rangle_R M_{+K'}\Phi^{-}_{QN} - N^{-}_{K'K} \left\langle M_{+K}\Phi^{-*}_{QN}, \hat{\Phi}\right\rangle_R M_{+K'}\Phi^{-*}_{QN}\Big),\end{aligned} \tag{5.3}$$

where the $N^{\pm}_{KK'}$ are defined through

$$\sum_{K'} N^{\pm}_{KK'} \left\langle M_{+K'}\Phi^{\pm}_{QN}, M_{+L}\Phi^{\pm}_{QN}\right\rangle_R = \delta_{KL}. \tag{5.4}$$

Then, using (5.2), the quasinormal mode Green function is given by

$$G(x;x') = \sum_{K,K'} \Phi^{+}_{K}(x)\Phi^{+*}_{K'}(Rx')N^{+}_{KK'} + \sum_{K,K'} \Phi^{-}_{K}(x)\Phi^{-*}_{K'}(Rx')N^{-}_{KK'}, \tag{5.5}$$

where $\Phi^{\pm}_{K} \equiv M_{+K}\Phi^{\pm}_{QN}$.

A demonstration that the function $G(x;x')$ so obtained is indeed the standard Euclidean Green function $G_E(x;x')$ implies that the quasinormal modes in (5.1) form a complete basis, in the sense that they satisfy

$$
\begin{aligned}
i\delta^3(\Omega-\Omega') &= \sqrt{\gamma}n^\mu \sum_{K,K'} N^+_{KK'} \left[\Phi^{+*}_K(t,\Omega)\nabla_\mu \Phi^+_{K'}(t,R\Omega') - \Phi^+_K(t,\Omega)\nabla_\mu \Phi^{+*}_{K'}(t,R\Omega')\right] \\
&\quad + (+\leftrightarrow -) \\
0 &= \sum_{K,K'} N^+_{KK'} \left[\Phi^+_K(t,\Omega)\Phi^{+*}_{K'}(t,R\Omega') - \Phi^{+*}_K(t,\Omega)\Phi^+_{K'}(t,R\Omega')\right] + (+\leftrightarrow -)\,,
\end{aligned} \tag{5.6}
$$

on a constant time slice with normal vector n^μ and induced metric $\gamma_{\mu\nu}$. Indeed, these two equations can be used to construct a retarded Green function, which in turn provides a solution to the wave equation with arbitrary initial data. Hence any suitably smooth solution to the wave equation can be decomposed on a Cauchy surface in terms of such a set of modes.

First, we would like to evaluate the sum (5.5) for the case $(x;x') = (t,\Omega_{SP};t',\Omega_{SP})$ where both points lie on the south pole observer's worldline. The functions $\Phi^\pm_K(t,\Omega_{SP})$ are nonzero only for spherically symmetric descendants $L^n_{+1}\Phi^\pm_{QN}(t,\Omega)$ where $L_{\mp 1} \equiv \sum_{k=1}^{3} M_{\mp k}M_{\mp k}$. The norm for such states is calculated in Appendix B and is given by

$$
\left\langle L^n_{+1}\Phi^\pm_{QN}, L^m_{+1}\Phi^\pm_{QN}\right\rangle_R = \frac{\Gamma(2+2n)\Gamma(2h_\pm+2n-1)}{\Gamma(2h_\pm-1)} \left\langle \Phi^\pm_{QN}, \Phi^\pm_{QN}\right\rangle_R \delta_{nm}, \tag{5.7}
$$

while the modes at $\Omega = \Omega_{SP}$ are given by

$$
\begin{aligned}
L^n_{+1}\Phi^-_{QN}(t,\Omega_{SP}) &= \frac{\Gamma(2n+2)}{2\pi}\frac{e^{-nt}}{(e^{+t}-i\epsilon)^{n+1}}, \\
L^n_{+1}\Phi^+_{QN}(t,\Omega_{SP}) &= -\frac{\Gamma(2n+3)}{2\sqrt{2}\pi}\frac{e^{-nt}}{(e^{+t}-i\epsilon)^{n+2}}.
\end{aligned} \tag{5.8}
$$

Using

$$
\begin{aligned}
\left(L^n_{+1}\Phi^-_{QN}(t,R\Omega_{SP})\right)^* &= -L^n_{+1}\Phi^-_{QN}(-t,\Omega_{SP}) \\
\left(L^n_{+1}\Phi^+_{QN}(t,R\Omega_{SP})\right)^* &= L^n_{+1}\Phi^+_{QN}(-t,\Omega_{SP}),
\end{aligned} \tag{5.9}
$$

the full sum (5.5) is

$$
\begin{aligned}
G(t,\Omega_{SP};t',\Omega_{SP}) &= -\frac{1}{4\pi^2}\sum_{k=0}^{\infty}\left\{\frac{(2k+1)e^{-k(t-t')}}{\left[(e^{+t}-i\epsilon)\left(e^{-t'}-i\epsilon\right)\right]^{k+1}} + \frac{(2k+2)e^{-k(t-t')}}{\left[(e^{+t}-i\epsilon)\left(e^{-t'}-i\epsilon\right)\right]^{k+2}}\right\} \\
&= -\frac{1}{16\pi^2}\frac{1}{\sinh^2[(t-t')/2]-i\epsilon\tilde{s}(x;x')},
\end{aligned} \tag{5.10}
$$

where

$$\tilde{s}(x;x') \equiv \frac{\sinh t - \sinh t'}{1 + e^{t'-t}}. \tag{5.11}$$

Noting that for small ϵ, $\tilde{s}(x;x')$ is equivalent to $s(x;x')$ defined in (2.18), it follows that this Green function agrees with that in (2.17) on the south pole observer's worldline. Since the construction of our Green function is dS-invariant[‡], agreement on this worldline implies that this Green function equals the Euclidean one on any two *timelike* separated points.

For *spacelike* separated points, we find from (5.5) that

$$G(t,\Omega_{SP};t',\Omega_{NP}) = \frac{1}{8\pi^2\left[1+\cosh\left(t+t'\right)\right]} = G_E(t,\Omega_{SP};t',\Omega_{NP}). \tag{5.12}$$

By dS-invariance, we can extend this to any two spacelike-separated points. This concludes the proof that the quasinormal Green function (5.5) is indeed the Euclidean Green function.

6 Results for general light scalars ($m^2\ell^2 < 9/4$)

In the general case of a light scalar with $m^2\ell^2 < 9/4$, we can write out the explicit form of the Euclidean two-point function as (see for instance [9])

$$G_E(x;x') = \frac{\Gamma(h_+)\Gamma(h_-)}{16\pi^2} F\left[h_+, h_-, 2, \frac{1+P(x;x') - is(x;x')\epsilon}{2}\right], \tag{6.1}$$

where

$$h_\pm = \frac{3}{2} \pm \mu, \qquad \mu = \sqrt{\frac{9}{4} - m^2\ell^2}. \tag{6.2}$$

The asymptotic behaviors of the Euclidean Green function are:

$$\begin{aligned}
\lim_{t'\to\infty} G_E(t,\Omega;t',\Omega_{NP}) &= \frac{\Gamma(h_- - h_+)\Gamma(h_+)}{2^{4-2h_+}\pi^2\Gamma(2-h_+)} \frac{e^{-h_+t'}}{(\sinh t - i\epsilon + \cosh t\cos\psi)^{h_+}} + (h_+ \leftrightarrow h_-), \\
\lim_{t'\to\infty} G_E(t,\Omega;-t',\Omega_{SP}) &= e^{-i\pi h_+}\frac{\Gamma(h_- - h_+)\Gamma(h_+)}{2^{4-2h_+}\pi^2\Gamma(2-h_+)} \frac{e^{-h_+t'}}{(\sinh t - i\epsilon + \cosh t\cos\psi)^{h_+}} \\
&\quad +(h_+ \leftrightarrow h_-).
\end{aligned} \tag{6.3}$$

Note that in dealing with the branch-cut of $G_E(t,\Omega;t',\Omega')$, we go under (above) it when $t > t'$ ($t < t'$) in accordance with the $i\epsilon$-prescription.

[‡]This follows from the fact that the Green function is just a position-space representation of the projection operator onto the highest-weight representation of the three-dimensional conformal group characterized by the highest-weight $-h$, as can be seen by writing out this projection as a sum over complete states of the representation and using the definition of $\mathsf{SO}(4,1)$ generators.

Let us define $G_\pm$ as

$$G_\pm(x;x') \equiv G_E(x;x') - e^{i\pi h_\mp} G_E(x;x'_A). \tag{6.4}$$

These satisfy the future boundary conditions in Ref. [21] in the region $P < -1$. Now, we define the highest-weight modes as

$$\Phi^\pm_{QN}(x) \equiv \lim_{t'\to\infty} e^{h_\pm t'} G_\pm(t,\Omega;t',\Omega_{NP}). \tag{6.5}$$

The $\Phi^\pm_{QN}$ are explicitly given by

$$\Phi^\pm_{QN}(x) = \frac{1}{4\pi^{5/2}} \frac{\Gamma(\mp\mu)\Gamma(h_\pm)\left(1-e^{\mp 2\pi i\mu}\right)}{[\sinh t - i\epsilon + \cosh t\cos\psi]^{h_\pm}} = \frac{1}{4\pi^{5/2}} \frac{\Gamma(\mp\mu)\Gamma(h_\pm)\left(1-e^{\mp 2\pi i\mu}\right)}{\cosh^{h_\pm} t\,[\tanh(t-i\epsilon)+\cos\psi]^{h_\pm}}. \tag{6.6}$$

The asymptotic behavior of the modes as $t\to\infty$ is

$$\lim_{t\to\infty}\Phi^\pm_{QN}(t,\Omega) = \frac{2^{3-h_\pm}}{\sqrt{\pi}}\Gamma(\mp\mu)\Gamma(h_\pm)\left(1-e^{\mp 2\pi i\mu}\right)\Delta_\pm(\Omega,\Omega_{NP})e^{-h_\pm t} \mp \frac{4i}{\mu}\frac{\delta^3(\Omega-\Omega_{NP})}{\sqrt{h}}e^{-h_\mp t}. \tag{6.7}$$

We have defined[§]

$$\begin{aligned}\Delta_\pm(\Omega,\Omega') &= \frac{2^{3(h_\pm-1)}}{\pi}\Gamma(2-2h_\pm)\sin(h_\pm\pi)\sum\frac{\Gamma(h_\pm+L)}{\Gamma(h_\mp+L)}Y_{Lj}(\Omega)Y^*_{Lj}(\Omega') \\ &= \frac{1}{2^{5-2h_\pm}\pi^2}\frac{1}{[1-\cos\Theta_3(\Omega,\Omega')]^{h_\pm}},\end{aligned} \tag{6.9}$$

which satisfy

$$\frac{\pi^2}{8\cos^2(\pi\mu)\Gamma(2-2h_+)\Gamma(2-2h_-)}\int d^3\Omega''\sqrt{h}\;\Delta_+(\Omega,\Omega'')\Delta_-(\Omega'',\Omega') = \frac{1}{\sqrt{h}}\delta^3(\Omega-\Omega'). \tag{6.10}$$

The norm is easily evaluated at $\mathcal{I}^+$ to be

$$\left\langle \Phi^\pm_{\Omega_1},\Phi^\pm_{\Omega_2}\right\rangle_R = \frac{2^{5-h_\pm}}{\sqrt{\pi}}\Gamma(\mp\mu)\Gamma(h_\pm)\sin^2(\pi\mu)\Delta_\pm(\Omega_1,R\Omega_2),$$

§Here, a useful identity on S^3 is

$$[1-\cos\Theta_3(\Omega,\Omega')]^{-h} = 2^{2+h}\pi\sin(\pi h)\Gamma(2-2h)\sum\frac{\Gamma(L+h)}{\Gamma(3+L-h)}Y_{Lj}(\Omega)Y^*_{Lj}(\Omega'). \tag{6.8}$$

$$\left\langle \Phi^{\pm}_{\Omega_1}, \Phi^{\mp}_{\Omega_2} \right\rangle_R = \pm\frac{4i}{\mu}\left(1-e^{\pm 2\pi i\mu}\right)\frac{\delta^3(\Omega_1 - R\Omega_2)}{\sqrt{h}}. \tag{6.11}$$

As such, we find that the R-norms of the quasinormal modes are

$$\left\langle \Phi^{\pm}_{QN}, \Phi^{\pm}_{QN} \right\rangle_R = \Gamma(\mp\mu)\Gamma(h_\pm)\frac{\sin^2(\pi\mu)}{\pi^{5/2}}, \qquad \left\langle \Phi^{+}_{QN}, \Phi^{-}_{QN} \right\rangle_R = 0. \tag{6.12}$$

The rest of the discussion on the induced norms of the descendants carries over from the $m^2\ell^2 = 2$ case.

Next, we follow our previous strategy of showing that the mode sum and the Euclidean Green function agree on the south pole observer's worldline. Again, we evaluate

$$L^n_{+1}\Phi^{\pm}_{QN}(t,\Omega_{SP}) = \frac{\Gamma(\mp\mu)(1-e^{\mp 2\pi i\mu})}{4\pi^{5/2}}\frac{\Gamma(2n+3)\Gamma(h_\pm + n)}{2\Gamma(n+2)}\frac{e^{-nt}}{(e^t - i\epsilon)^{n+h_\pm}}. \tag{6.13}$$

As before, the norm for such states is

$$\left\langle L^n_{+1}\Phi^{\pm}_{QN}, L^n_{+1}\Phi^{\pm}_{QN} \right\rangle_R = \frac{\Gamma(2+2n)\Gamma(2h_\pm + 2n - 1)}{\Gamma(2h_\pm - 1)}\left\langle \Phi^{\pm}_{QN}, \Phi^{\pm}_{QN} \right\rangle_R. \tag{6.14}$$

Note that

$$\left(L^n_{+1}\Phi^{\pm}_{QN}(t, R\Omega_{SP})\right)^* = e^{i\pi h_\pm}L^n_{+1}\Phi^{\pm}_{QN}(-t,\Omega_{SP}). \tag{6.15}$$

The quasinormal mode Green function (5.5) is then

$$G(t,\Omega_{SP};t',\Omega_{SP}) = \sum_{n=0}^{\infty}\left[\frac{c_n^+ e^{-n(t-t')}}{\left[(e^t - i\epsilon)\left(e^{-t'} - i\epsilon\right)\right]^{n+h_+}} + \frac{c_n^- e^{-n(t-t')}}{\left[(e^t - i\epsilon)\left(e^{-t'} - i\epsilon\right)\right]^{n+h_-}}\right] \tag{6.16}$$

where

$$c_n^{\pm} = -\frac{e^{-i\pi h_\pm}}{2\pi^2\sin(\pm\pi\mu)}\frac{\Gamma(\frac{3}{2}+n)\Gamma(h_\pm + n)}{\Gamma(1+n\pm\mu)\Gamma(1+n)}. \tag{6.17}$$

In Appendix C, we show that on the south pole observer's worldline, this Green function is equal to the Euclidean Green function. Thus by dS-invariance of both Green functions, they agree for any two timelike separated points in dS_4.

For spacelike separated points, we consider the Euclidean Green function with one point at the south pole and the other point at the north pole. One notices that

$$G_E(t,\Omega_{SP};t',\Omega_{NP}) = G_E(t,\Omega_{SP};t',\Omega_{SP})|_{t'\to -t'+i\pi}. \tag{6.18}$$

Since these points are spacelike separated, we do not have to worry about the $i\epsilon$-prescription and the Green function is real. If we had evaluated the quasinormal mode Green function $G(t,\Omega_{SP};t',\Omega_{NP})$, then we would have obtained the same sum as in (6.16), provided that we sent $t'\to -t'$ and removed the phase $e^{-i\pi h_\pm}$ from the coefficients (6.17). This is equivalent to sending $t'\to -t'+i\pi$ and hence by dS-invariance we have proved that for any two *spacelike* separated points, the quasinormal mode Green function is the Euclidean Green function.

7 Southern modes and T-norm

In this section we find quasinormal modes that vanish in the northern or southern diamonds – the analogues of Rindler modes in Minkowski space. We begin with the expression (6.6) for the lowest-weight mode

$$\Phi^{\pm}_{QN}(x)=\frac{1}{4\pi^{5/2}}\frac{\Gamma(\mp\mu)\Gamma(h_\pm)\left(1-e^{\mp 2\pi i\mu}\right)}{\left[\sinh t-i\epsilon+\cosh t\cos\psi\right]^{h_\pm}}. \tag{7.1}$$

For generic mass $\Phi^{\pm}_{QN}(x)$ has a branch cut on the past horizon of the southern observer at $\tanh t=-\cos\psi$. We have chosen the phase convention so that the denominator is real above the past horizon. Crossing the past horizon gives an extra phase of $e^{i\pi h_\pm}$. It follows that the southern mode

$$\Phi^{\pm}_{QN,S}(x)\equiv\Phi^{\pm}_{QN}(x)+\Phi^{\pm *}_{QN}(x) \tag{7.2}$$

vanishes below the past horizon. Similarly the northern mode

$$\Phi^{\pm}_{QN,N}(x)\equiv e^{i\pi h_\pm}\Phi^{\pm}_{QN}(x)+e^{-i\pi h_\pm}\Phi^{\pm *}_{QN}(x). \tag{7.3}$$

vanishes above the past horizon. The R-norms between these modes are

$$\begin{aligned}
\left\langle\Phi^{\pm}_{QN,S},\Phi^{\pm}_{QN,S}\right\rangle_R &= \left\langle\Phi^{\pm}_{QN,N},\Phi^{\pm}_{QN,N}\right\rangle_R = \left\langle\Phi^{\pm}_{QN,N},\Phi^{\mp}_{QN,N}\right\rangle_R = \left\langle\Phi^{\mp}_{QN,S},\Phi^{\mp}_{QN,S}\right\rangle_R = 0,\\
\left\langle\Phi^{\pm}_{QN,N},\Phi^{\pm}_{QN,S}\right\rangle_R &= -2i\sin(\pi h_\pm)\left\langle\Phi^{\pm}_{QN},\Phi^{\pm}_{QN}\right\rangle_R = -2i\sin(\pi h_\pm)\Gamma(\mp\mu)\Gamma(h_\pm)\frac{\sin^2(\pi\mu)}{\pi^{5/2}}.
\end{aligned}\tag{7.4}$$

On the other hand, the R-norm for the global quasinormal modes is closely related to time-reflection:

$$\left\langle f,\Phi^{\pm}_K\right\rangle_R=e^{-i\pi h_\pm}\left\langle f,T\Phi^{\pm *}_K\right\rangle_{KG},\qquad T:t\to -t. \tag{7.5}$$

While the R-norm has no analogue in the static patch, the T-norm is easily generalizable to the southern diamond as

$$\left\langle \Phi^{\pm}_{K}, \Phi^{\pm}_{K'} \right\rangle_{T,B^3_S} \equiv \left\langle T\Phi^{\pm *}_{K}, \Phi^{\pm}_{K'} \right\rangle_{KG,B^3_S}, \tag{7.6}$$

where B^3_S denotes the integral over a complete slice in the southern diamond. We have

$$\begin{aligned} \left\langle \Phi^{\pm}_{QN,S}, T\Phi^{\pm}_{QN,S} \right\rangle_{T,B^3_S} &= \left\langle \Phi^{\pm}_{QN,S}, T\Phi^{\mp}_{QN,S} \right\rangle_{T,B^3_S} = \left\langle \Phi^{\pm}_{QN,S}, \Phi^{\mp}_{QN,S} \right\rangle_{T,B^3_S} = 0, \\ \left\langle \Phi^{\pm}_{QN,S}, \Phi^{\pm}_{QN,S} \right\rangle_{T,B^3_S} &= 2i \sin(\pi h_{\pm}) \Gamma(\mp\mu) \Gamma(h_{\pm}) \frac{\sin^2(\pi\mu)}{\pi^{5/2}}. \end{aligned} \tag{7.7}$$

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A Appendix: dS_4 Killing vectors

In global coordinates, the 10 Killing vectors of dS_4 are given by:

$$\begin{aligned} L_0 &= \cos\psi\,\partial_t - \tanh t \sin\psi\,\partial_\psi, \\ M_{\mp 1} &= \pm \sin\psi \sin\theta \sin\phi\,\partial_t + (1 \pm \tanh t \cos\psi) \sin\theta \sin\phi\,\partial_\psi \\ &\quad + (\cot\psi \pm \tanh t \csc\psi)(\cos\theta \sin\phi\,\partial_\theta + \csc\theta \cos\phi\,\partial_\phi), \\ M_{\mp 2} &= \pm \sin\psi \sin\theta \cos\phi\,\partial_t + (1 \pm \tanh t \cos\psi) \sin\theta \cos\phi\,\partial_\psi \\ &\quad + (\cot\psi \pm \tanh t \csc\psi)(\cos\theta \cos\phi\,\partial_\theta - \csc\theta \sin\phi\,\partial_\phi), \\ M_{\mp 3} &= \pm \sin\psi \cos\theta\,\partial_t + (1 \pm \tanh t \cos\psi) \cos\theta\,\partial_\psi - (\cot\psi \pm \tanh t \csc\psi) \sin\theta\,\partial_\theta, \\ J_1 &= \cos\phi\,\partial_\theta - \sin\phi \cot\theta\,\partial_\phi, \\ J_2 &= -\sin\phi\,\partial_\theta - \cos\phi \cot\theta\,\partial_\phi, \\ J_3 &= \partial_\phi. \end{aligned} \tag{A.1}$$

Their non-zero commutators are:

$$\begin{aligned}
[J_i, J_j] &= \sum_{k=1}^{3} \epsilon_{ijk} J_k, \qquad & [J_i, M_{\pm j}] &= \sum_{k=1}^{3} \epsilon_{ijk} M_{\pm k}, \\
[L_0, M_{\pm i}] &= \mp M_{\pm i}, \qquad & [M_{+i}, M_{-j}] &= 2L_0 \delta_{ij} + 2\sum_{k=1}^{3} \epsilon_{ijk} J_k.
\end{aligned} \tag{A.2}$$

As expected, these are the relations which define the $\mathsf{SO}(4,1)$ algebra. The commutators on the first line indicate that the J_i generate an $\mathsf{SO}(3)$ subalgebra, under which the M_{+i} and the M_{-i} transform as vectors. The second line implies that for each $i \in \{1,2,3\}$, the Killing vectors $M_{\pm i}$ and L_0 form an $\mathsf{SO}(2,1)$ subalgebra satisfying (not summing over i)

$$[M_{+i}, M_{-i}] = 2L_0, \qquad [L_0, M_{\pm i}] = \mp M_{\pm i}. \tag{A.3}$$

The scalar Laplacian is a Casimir operator. It reads:

$$\ell^2 \nabla^2 = -L_0(L_0 - 3) + \sum_{i=1}^{3} M_{-i} M_{+i} + J^2. \tag{A.4}$$

The convention is that $J^2 = -L(L+1)$ on the spherical harmonics Y_{Lj}. The conformal Killing vectors of the S^3 are given by the restriction of dS_4 Killing vectors on $\mathcal{I}^+$:

$$\begin{aligned}
L_0 &= -\sin\psi\,\partial_\psi, \\
M_{\mp 1} &= (1 \pm \cos\psi)\sin\theta\sin\phi\,\partial_\psi + (\cot\psi \pm \csc\psi)(\cos\theta\sin\phi\,\partial_\theta + \csc\theta\cos\phi\,\partial_\phi), \\
M_{\mp 2} &= (1 \pm \cos\psi)\sin\theta\cos\phi\,\partial_\psi + (\cot\psi \pm \csc\psi)(\cos\theta\cos\phi\,\partial_\theta - \csc\theta\sin\phi\,\partial_\phi), \\
M_{\mp 3} &= (1 \pm \cos\psi)\cos\theta\,\partial_\psi - (\cot\psi \pm \csc\psi)\sin\theta\,\partial_\theta, \\
J_1 &= \cos\phi\,\partial_\theta - \sin\phi\cot\theta\,\partial_\phi, \\
J_2 &= -\sin\phi\,\partial_\theta - \cos\phi\cot\theta\,\partial_\phi, \\
J_3 &= \partial_\phi.
\end{aligned} \tag{A.5}$$

To relate the above de Sitter generators to the embedding coordinates X defined by

$$\eta_{\mu\nu} X^\mu X^\nu = \ell^2, \tag{A.6}$$

where η has signature $(4,1)$ and the usual Lorentz generators are given by

$$M_{\mu\nu} = X_\mu\,\partial_\nu - X_\nu\,\partial_\mu, \tag{A.7}$$

with commutators

$$[M_{\mu\nu}, M_{\alpha\beta}] = \eta_{\alpha\nu} M_{\mu\beta} - \eta_{\alpha\mu} M_{\nu\beta} - \eta_{\beta\nu} M_{\mu\alpha} + \eta_{\beta\mu} M_{\nu\alpha}, \tag{A.8}$$

we have for $i, j, k \in \{1, 2, 3\}$

$$\begin{aligned} L_0 &= M_{40}, \\ M_{\mp k} &= M_{4k} \mp M_{0k}, \\ J_i &= -\epsilon_{ijk} M_{jk}. \end{aligned} \tag{A.9}$$

The standard Klein-Gordon adjoint acts as:

$$M^\dagger_{\mu\nu} = -M_{\mu\nu}, \tag{A.10}$$

where the adjoint is defined in the standard way as $\left\langle f, M^\dagger g \right\rangle_{KG} \equiv \langle M f, g \rangle_{KG}$. The action of R on the Killing vectors is:

$$L_0 \to -L_0, \qquad J_k \to J_k, \qquad M_{\pm k} \to -M_{\mp k}, \tag{A.11}$$

or equivalently, for $j, k \in \{1, 2, 3\}$,

$$M_{40} \to -M_{40}, \qquad M_{4k} \to -M_{4k}, \qquad M_{0k} \to M_{0k}, \qquad M_{jk} \to M_{jk}. \tag{A.12}$$

In the R-norm, if we define $M^{\dagger_R}$ as $\left\langle f, M^{\dagger_R} g \right\rangle_R \equiv \langle M f, g \rangle_R$ then

$$M^{\dagger_R}_{40} = M_{40}, \qquad M^{\dagger_R}_{4k} = M_{4k}, \qquad M^{\dagger_R}_{0k} = -M_{0k}, \qquad M^{\dagger_R}_{jk} = -M_{jk}. \tag{A.13}$$

With respect to the R-norm, the antihermitian generators are iM_{40}, iM_{4k}, M_{0k} and M_{jk}. On the other hand, we have from (A.8) that for $i, j, k \in \{1, 2, 3\}$,

$$\begin{aligned} &[M_{0i}, M_{0j}] = -\eta_{00} M_{ij}, \quad [M_{0i}, M_{04}] = -\eta_{00} M_{i4}, \quad [M_{0i}, M_{j4}] = \eta_{ij} M_{04}, \\ &[M_{0i}, M_{jk}] = \eta_{ij} M_{0k}, \qquad [M_{04}, M_{j4}] = -\eta_{44} M_{0j}, \quad [M_{j4}, M_{k4}] = -\eta_{44} M_{jk}, \end{aligned} \tag{A.14}$$

while the M_{jk} obey the usual $\mathsf{SO}(3)$ algebra. Now, if we sent $M_{40} \to iM_{40}$ and $M_{4k} \to iM_{4k}$, we would get the same algebra but with $\eta_{44} \to -\eta_{44}$. This demonstrates that the insertion of R in the norm transforms $\mathsf{SO}(4,1)$ into $\mathsf{SO}(3,2)$.

B Appendix: Norm for spherically symmetric states

Consider the operator $L_{\mp 1} \equiv \sum_{k=1}^{3} M_{\mp k} M_{\mp k}$, which evidently satisfies $[J_k, L_{\mp 1}] = 0$. Defining $|h+n\rangle \equiv L_{+1}^n |h\rangle$, where $|h\rangle$ is the spherically symmetric highest-weight state with $J^2 |h\rangle = 0$ and $L_0 |h\rangle = -h |h\rangle$, we have

$$[L_{+1}, L_{-1}] |h+n\rangle = 4L_0 \left(2L_0^2 + 2\nabla^2 - 3\right) |h+n\rangle . \tag{B.1}$$

The Casimir is

$$\nabla^2 = -L_0(L_0 - 3) + M_{-k}M_{+k} + J^2 = -L_0(L_0+3) + M_{+k}M_{-k} + J^2 \tag{B.2}$$

and $\nabla^2 |h+n\rangle = -h(h-3) |h+n\rangle$. Then using

$$[L_{-1}, L_{+1}] |h+n\rangle = 4(h+2n)(8n^2 + 8nh + 6h - 3) |h+n\rangle , \tag{B.3}$$

it is straightforward to show that

$$\begin{aligned} \langle h| L_{-1}^n L_{+1}^n |h\rangle &= 4n(n+h-1)(2n+1)(2n+2h-3) \langle h| L_{-1}^{n-1} L_{+1}^{n-1} |h\rangle \\ &= \frac{\Gamma(2+2n)\Gamma(2h+2n-1)}{\Gamma(2h-1)} \langle h|h\rangle . \end{aligned} \tag{B.4}$$

C Appendix: Green function at the south pole

We wish to evaluate the sum (5.5) over the quasinormal modes for the massive case,

$$\begin{aligned} G(t, \Omega_{SP}; t', \Omega_{SP}) &= \left[\left(e^t - i\epsilon\right)\left(e^{-t'} - i\epsilon\right)\right]^{-h_+} \sum_n c_n^+ \left[\frac{e^{-(t-t')}}{(e^t - i\epsilon)(e^{-t'} - i\epsilon)}\right]^n \\ &\quad + \left[\left(e^t - i\epsilon\right)\left(e^{-t'} - i\epsilon\right)\right]^{-h_-} \sum_n c_n^- \left[\frac{e^{-(t-t')}}{(e^t - i\epsilon)(e^{-t'} - i\epsilon)}\right]^n \end{aligned} \tag{C.1}$$

where

$$c_n^{\pm} = \pm \frac{e^{-i\pi h_\pm}}{(-2\pi^2)\sin(\pi\mu)} \frac{\Gamma(\frac{3}{2}+n)\Gamma(h_\pm + n)}{\Gamma(1+n\pm\mu)\Gamma(1+n)}. \tag{C.2}$$

Each sum combines into a hypergeometric function with argument shifted by ϵ

$$\begin{aligned} &2\sin(\pi\mu) G(t, \Omega_{SP}; t', \Omega_{SP}) \\ &= \frac{e^{-i\pi h_+}}{[(e^t - i\epsilon)(e^{-t'} - i\epsilon)]^{h_+}} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(h_+)}{(-\pi^2)\Gamma(1+\mu)} F\left[h_+, \frac{3}{2}; 1+\mu; \frac{e^{-(t-t')}}{(e^t - i\epsilon)(e^{-t'} - i\epsilon)}\right] \end{aligned}$$

$$- \quad \frac{e^{-i\pi h_-}}{\left[(e^t - i\epsilon)\left(e^{-t'} - i\epsilon\right)\right]^{h_-}} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(h_-)}{(-\pi^2)\Gamma(1-\mu)} F\left[h_-, \frac{3}{2}; 1-\mu; \frac{e^{-(t-t')}}{(e^t - i\epsilon)\left(e^{-t'} - i\epsilon\right)}\right]. \quad \text{(C.3)}$$

Kummer's quadratic transformation

$$F\left[\alpha, \beta, 2\beta, \frac{4z}{(1+z)^2}\right] = (1+z)^{2\alpha} F\left[\alpha, \alpha - \beta + \frac{1}{2}, \beta + \frac{1}{2}, z^2\right] \quad \text{(C.4)}$$

with

$$z \equiv \frac{e^{-(t-t')/2}}{\left[(e^t - i\epsilon)\left(e^{-t'} - i\epsilon\right)\right]^{1/2}} \quad \text{(C.5)}$$

allows us to rewrite the hypergeometric functions in the more recognizable form

$$2\sin(\pi\mu) G(t, \Omega_{SP}; t', \Omega_{SP}) = e^{-i\pi h_+} R^{-h_+} H_+(P_\epsilon) - e^{-i\pi h_-} R^{-h_-} H_-(P_\epsilon) \quad \text{(C.6)}$$

where $H_\pm$ are analytical continuations of the Green functions $G_\pm$ we defined in (6.4) to the region $P > 1$:

$$H_\pm(x; x') = G_E(x; x') - e^{-i\pi h_\mp} G_E(x; x'_A). \quad \text{(C.7)}$$

They are explicitly given by

$$H_\pm(P_\epsilon) = \frac{\Gamma(\mp\mu)\Gamma(h_\pm)\sin(\pm\pi\mu)}{2^{1+2h_\pm}\pi^{5/2}} \left(\frac{2}{1+P}\right)^{h_\pm} F\left[h_\pm, h_\pm - 1; 2(h_\pm - 1), \frac{2}{1+P_\epsilon}\right] \quad \text{(C.8)}$$

with the argument

$$\frac{2}{1+P_\epsilon} \equiv \frac{4z}{(1+z)^2} \quad \text{(C.9)}$$

while R is some correction factor

$$R \equiv z\left(e^t - i\epsilon\right)\left(e^{-t'} - i\epsilon\right) = e^{-(t-t')/2}\left(e^t - i\epsilon\right)^{1/2}\left(e^{-t'} - i\epsilon\right)^{1/2}. \quad \text{(C.10)}$$

Note that when $\epsilon \to 0$, this correction $R \to 1$ and $z \to e^{-(t-t')}$.

Also, observe that (C.9) implies that

$$P_\epsilon = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left\{\frac{e^{-(t-t')/2}}{\left[(e^t - i\epsilon)\left(e^{-t'} - i\epsilon\right)\right]^{1/2}} + \frac{\left[(e^t - i\epsilon)\left(e^{-t'} - i\epsilon\right)\right]^{1/2}}{e^{-(t-t')/2}}\right\}. \quad \text{(C.11)}$$

Away from the singularity at $t = t'$, we can set ϵ to zero so that

$$G(t, \Omega_{SP}; t', \Omega_{SP}) = \frac{e^{-i\pi h_+} H_+(P) - e^{-i\pi h_-} H_-(P)}{2\sin(\pi\mu)} = G_E(t, \Omega_{SP}; t', \Omega_{SP}). \quad \text{(C.12)}$$

The singularity structure for G can be also analyzed from (C.6). The correction factor R is regular near the singularity, while the $G_{\pm}(P)$ have poles when P approaches 1. Expanding (C.11) to first order in ϵ yields

$$P_\epsilon = \cosh(t-t') - i\epsilon \sinh(t-t') \left(\frac{e^{-t} + e^{t'}}{2} \right) = P - i\epsilon \hat{s}(x, x') \tag{C.13}$$

with

$$\hat{s}(x, x') \equiv \sinh(t-t') \left(\frac{e^{-t} + e^{t'}}{2} \right). \tag{C.14}$$

The singularity structure therefore is the same as in definition (6.1) for the Euclidean Green function up to redefinition of ϵ by some positive function.

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